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Branched matrix models and the scales of supersymmetric gauge theories

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Abstract: In the framework of the matrix model/gauge theory correspondence, we consider supersymmetric $\mathrm{U}(N)$ gauge theory with $\mathrm{U}(1)^N$ symmetry breaking pattern. Due to the presence of the Veneziano–Yankielowicz effective superpotential, in order to satisfy the F–term condition $\sum_i S_i = 0$, we are forced to introduce additional terms in the free energy of the corresponding matrix model with respect to the usual formulation. This leads to a matrix model formulation with a cubic potential which is free of parameters and displays a branched structure. In this way we naturally solve the usual problem of the identification between dimensionful and dimensionless quantities. Furthermore, we need not introduce the $\mathcal{N} = 1$ scale by hand in the matrix model. These facts are related to remarkable coincidences which arise at the critical point and lead to a branched bare coupling constant. The latter plays the role of the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ scales tuning parameter. We then show that a suitable rescaling leads to the correct identification of the $\mathcal{N} = 2$ variables. Finally, by means of the mentioned coincidences, we provide a direct expression for the $\mathcal{N} = 2$ prepotential, including the gravitational corrections, in terms of the free energy. This suggests that the matrix model provides a triangulation of the istanton moduli space.

Keywords: Supersymmetric Effective Theories, String Duality.

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Contents

1. Introduction

During the last year, our understanding of the nonperturbative dynamics of fourdimensional supersymmetric gauge theory has achieved a dramatic advance. Motivated by insights from the geometric engineering perspective [1, 2], in a series of papers [3] Dijkgraaf and Vafa have proposed that some exact holomorphic quantities of $\mathcal{N}=1$ supersymmetric gauge theories are captured by an auxiliary matrix model. In particular, under the assumption that the low energy F-term physics is described by a glueball superfield, they proposed that the $\mathcal{N}=1$ effective superpotential is completely obtained by evaluating the genus zero free energy of the related matrix model. This conjecture has been proved by two different techniques, first by showing that the relevant superspace diagrams reduce to a zero dimensional theory [4] and then by showing that the generalized Konishi anomaly equations in the chiral ring of the gauge theory are equivalent to the loop equations of the related matrix model [5]–[8]. Very recently, an apparent discrepancy has been found between the standard field theory computation and the matrix model result, both in the perturbative approach [9] and from the Konishi anomaly point of view [10]. The solution of this puzzle has been proposed in [11] by investigating the ambiguity in the UV completion of the supersymmetric gauge theories.

A strong check of the DV conjecture can be performed in the cases where the gauge theory exact quantities are already available, e.g. by testing the well known $\mathcal{N}=2$ supersymmetric gauge theory, namely Seiberg-Witten theory [12]. A recent step in this direction [13] concerns the exploration of the way in which the well known duality structure of the SW theory appears inside the matrix model itself (see also [14, 15, 16]). The usual matrix model formulation of the SW theory [17] poses a number of questions. Let us briefly discuss the main issues.

The first concerns the extremization of the superpotential. The matrix model conjecture for the SW theory with $\mathrm{U}(N)$ gauge group requires the soft breaking of $\mathcal{N}=2$ to $\mathcal{N}=1$ by adding a tree level superpotential. Now, due to the usual structure of the Veneziano-Yankielovicz effective superpotential [18], namely the appearance of the log terms, the F-term condition for the glueball superfields S_i, strictly speaking, cannot give the expected extremum condition $\sum_{i=1}^{N} S_i = 0$.

A second crucial point is related to the fact that the most basic feature of the SW gauge theory, namely the duality structure, is not displayed in its matrix model counterpart. As explained in [13], the first step in investigating such deep aspect is to consider the scaling properties of the matrix model free energy. In particular, it was shown that the natural variable in order to display the SW duality is a rescaled version of the glueball superfield. Therefore, the question arise whether there exists a formulation of the conjecture that by itself provides this additional structure.

Another important issue is related to the introduction of dimensionful quantities in the matrix model. On one hand, in the usual formulation one introduces by hand a cutoff Λ directly in the free energy of the matrix model, relying on a gauge theory expectation. The presence of such a dimensionful object in the matrix model provides in turn serious troubles in analyzing the monodromy properties of the free energy. Moreover, the identification of the u-modulus of the SW theory and of the dynamically generated scale of the gauge theory is performed at the critical point by comparing it to the known SW curve. In this respect, one might ask if any information about the scales and the modulus of the gauge theory is present even outside the critical point. Finally, in the usual approach one identifies the dimensionless 't Hooft couplings of the matrix model with the dimensionful gauge theory glueball superfields. This identification then leads to some problems in the interpretation of the extremum condition, namely introducing the concept of "eigenvalue holes" corresponding to the unstable extrema of the matrix model potential. Actually, all the issues considered so far will turn out to be tightly related one to each other.

In this paper we will present a first investigation of this fact, mainly stating the most interesting results. The details of the calculations and crucial generalizations will be given in [19]. In section 2, by addressing the problem of the minimization of the superpotential, we will show that

$$\mathcal{F}_0(-S_2,-S_1) \neq \mathcal{F}_0(S_1,S_2)\,.$$

Requiring that this symmetry exactly holds, one should modify the free energy by adding some bilinear terms such that the new free energy

$$\mathcal{F}_0^{(k)} = \mathcal{F}_0 + \delta\mathcal{F}_0^{(k)}\,,$$

displays different branches that depend on the odd number $2k+1$. In this way we obtain

$$\mathcal{F}_0^{(k)}(e^{i\kappa}S_2, e^{-i\kappa}S_1) = \mathcal{F}_0^{(k)}(S_1,S_2)\,,$$

where $\kappa \equiv (2k+1)\pi$. Then we will show that, in order to compare the matrix model quantities with the well known SW exact results, we have to perform a rescaling transformation on the matrix model variables [13]. In section 3 we show that the proposed free energy is given by the matrix model with potential

$$W^{(k)}(\Phi) = \mathrm{Tr}\left(\frac{1}{2}e^{\frac{i}{2}\kappa}\Phi^2 + \frac{1}{3}e^{\frac{3i}{4}\kappa}\Phi^3 - \frac{\mathbf{1}}{12}\right),$$

where, with respect to the usual formulation, the couplings disappear. A crucial term in the evaluation of the matrix model is the gaussian contribution which, due to the phases, will be given by

$$e^{-\frac{i}{4}\kappa(M_1^2-M_2^2)}.$$

This will also solve the questions related to the identification of dimensionful quantities. We then show that, in order to reproduce the expected extremum condition, the correct gauge theory bare coupling is

$$\tau_0 \longrightarrow \tau_0^{(k)} = \frac{2}{\pi i}\ln\frac{\Lambda_2}{\Lambda_1} - \frac{\kappa}{2\pi},$$

where Λ_1 and Λ_2 are the dynamically generated scale of the $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetric gauge theories, respectively. This clearly shows the role of the bare coupling as the $\mathcal{N}=1$ and $\mathcal{N}=2$ scales tuning parameter. Moreover, the nonperturbative relation found in [13] implies that at the critical point

$$\Lambda_1^2 = 4u,$$

where u is the SW modulus. Finally, it turns out that the free energy directly evaluated at the critical point is equal, up to a numerical factor, to the one obtained by integrating twice the effective coupling constant evaluated at the extremum. In other words, we find that

$$e^{\hat{\mathcal{F}}/\Lambda_2^6} = \frac{1}{\mathrm{Vol}(\mathrm{U}(M_1)) \times \mathrm{Vol}(\mathrm{U}(M_2))}\int \mathcal{D}\Phi_1\mathcal{D}\Phi_2 e^{-W^{(k)}(\Phi_1,\Phi_2)}|_{M_1=e^{i\kappa}M_2\equiv\mathcal{S}/\Lambda_2^3}.$$

This in turn provides a direct matrix model formulation for the $\mathcal{N}=2$ instantons, including the gravitational corrections, and shows that the matrix model provides a kind of triangulation of the instanton moduli space.

2. Branches from the symmetry of the free energy

In this section we start investigating the structure of the planar contribution to the free energy $\mathcal{F}_0$ of the matrix model. Since the matrix model formulation reproducing the CIV prepotential [1] has the advantage of a direct test with the established results of SW theory, we focus on this case. Nevertheless, many of the results we will find have a more general validity as they concern the matrix model formulation by itself rather than the specific SW realization.

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Let us then start with the free energy derived in [17]. The strategy will be to analyze its structure in relation with the exact $\mathcal{N}=2$ results it should reproduce at the extremum. In the analysis we will use the relationship between the $\mathcal{N}=2$ u-modulus and the prepotential for the gaugino condensate derived in [13]. This investigation will lead to introduce additional terms to $\mathcal{F}_0$ that will help us in deriving the matrix model formulation.

2.1 Stating the problem

In our explicit computation we will consider the case of a U(2) gauge group spontaneously broken to U(1) $\times$ U(1) with a cubic tree level superpotential. In this simple case we have two superfields S_1 and S_2, that will describe the effective abelian dynamics. Let us consider the matrix model and write down the expression of the free energy [17]

$$\mathcal{F}_0(S_i) = \sum_{j=1,2} \frac{S_j^2}{2} \ln \frac{S_j}{\Delta^3} - (S_1+S_2)^2 \ln \frac{\Lambda}{\Delta} + \Delta^6 \sum_{n\geq 3} \sum_{j=0}^{n} c_{n,j} \left(\frac{S_1}{\Delta^3}\right)^{n-j} \left(\frac{S_2}{\Delta^3}\right)^{j} . \qquad (2.1)$$

It turns out that the coefficients of the expansion satisfy the property

$$c_{n,j} = (-1)^n c_{n,n-j} , \qquad c_{n,j} = (-1)^j |c_{n,j}| . \qquad (2.2)$$

Eq. (2.1) has been derived in [17] by the matrix model formulation, except for the term depending on Λ that should be added by hand, as expected from the gauge theory. This expression for $\mathcal{F}_0$ differs by the relative sign between the infinite sum and the first two contributions with respect to [17] (as we will see, this fits with the implied expressions for the $\mathcal{N}=2$ modulus u and the effective coupling constant τ). Note that besides the second term, also the third one, as follows by $c_{n,j}=(-1)^n c_{n,n-j}$, is symmetric under the transformation $S_1 \to -S_2$, $S_2 \to -S_1$. However, due to the log term, we have

$$\mathcal{F}_0(-S_2,-S_1) \neq \mathcal{F}_0(S_1,S_2) .$$

On the other hand, to get the value $\langle S_1\rangle = -\langle S_2\rangle \equiv S$ at the extremum we need the exact symmetry. The extremum corresponds to the minimum of W_{eff}, which is usually evaluated by setting the bare coupling constant τ_0 to zero. This gives

$$\sum_i \tau_{ij} = 0 , \qquad j=1,2 , \qquad (2.3)$$

where $\tau_{ij} = \frac{\partial^2 \mathcal{F}_0}{\partial S_i \partial S_j}$, that is $\tau_{11}=\tau_{22}=-\tau_{12}$. On the other hand, setting $S_1=-S_2$, in τ_{ij}, gives

$$\tau_{11}-\tau_{22} = \ln\left(\frac{S_1}{\Delta^3}\right) - \ln\left(-\frac{S_1}{\Delta^3}\right) = (2k+1)\pi i, \qquad (2.4)$$

$k \in \mathbb{Z}$, so that $\tau_{11}-\tau_{22}$ does not vanish. Therefore, the symmetry of $\mathcal{F}_0$ under $S_1 \to -S_2$, $S_2 \to -S_1$ should be exact in order to get the critical value $S_2=-S_1$. This suggests modifying $\mathcal{F}_0$ in such a way that the following two crucial features hold:

1) The critical value for S as a function of Δ and Λ, which follows from the condition $\tau_{12}+\tau_{11}=0$ evaluated at the extremum, be unchanged and fit with the exact result [17, 13].

2) The effective gauge coupling constant τ_{11} evaluated at the critical point reproduces the well known SW exact result [12].

It turns out that if one chooses a vanishing bare coupling constant, then there is a modification to the free energy satisfying the above conditions, except for an apparently irrelevant term. The additional term reads

$$\delta\mathcal{F}_0^{(k)} = \frac{\Delta^3}{12}(S_1 - S_2) - \frac{i}{4}\kappa(S_1^2 - S_2^2) + \frac{i}{2}\kappa S_1 S_2 - \frac{3}{4}(S_1^2 + S_2^2)\,, \tag{2.5}$$

where

$$\kappa \equiv (2k+1)\pi\,, \qquad k \in \mathbb{Z}\,,$$

so that the modified free energy $\mathcal{F}_0^{(k)} \equiv \mathcal{F}_0 + \delta\mathcal{F}_0^{(k)}$ displays the requested symmetry

$$\mathcal{F}_0^{(k)}(e^{i\kappa}S_2, e^{-i\kappa}S_1) = \mathcal{F}_0^{(k)}(S_1, S_2)\,. \tag{2.6}$$

In the following, after discussing the crucial scaling properties of the free energy, we will check that the addition of (2.5) to the free energy reproduces the requested features at the extremum (see also [19]). However, we will see that it remains a “minor discrepancies”. Removing it will lead to the exact formulation with a unequivocally fixed bare coupling constant.

2.2 Rescaling the free energy

In [13] it has been shown that the free energy satisfies a scaling property which selects the natural variables to make duality transparent. In this respect, we note that the duality one obtains in $\mathcal{N}=1$ is the one induced, by consistency, by the $\Gamma(2)$ monodromy of $\mathcal{N}=2$. The scaling property of the free energy is obtained by first rescaling S_i, Δ and Λ

$$S_i \longrightarrow \mathcal{S}_i = \left(\frac{\Lambda}{\Delta}\right)^3 S_i\,, \qquad \Delta \longrightarrow \frac{\Lambda}{\Delta}\Delta = \Lambda\,, \qquad \Lambda \longrightarrow \frac{\Lambda}{\Delta}\Lambda = \frac{\Lambda^2}{\Delta}\,, \tag{2.7}$$

and then performing the map

$$\mathcal{F}_0^{(k)}(S_i, \Delta, \Lambda) \longrightarrow \mathcal{F}_0^{(k)}(\mathcal{S}_i, \Lambda, \mu\Lambda) = \mu^6 \mathcal{F}_0^{(k)}(S_i, \Delta, \Lambda)\,, \tag{2.8}$$

where $\mu \equiv \frac{\Lambda}{\Delta}$. Note that since the comparison with the SW curve gives [17] $\Delta^2 = 4u$, we thus have $\mu = (\Lambda^2/4u)^{1/2}$. We observe that whereas in the original free energy the scale Λ appears in pair with Δ, in the rescaled free energy we have that μ is “decoupled” from Λ. More precisely, $\mathcal{F}_0^{(k)}$ has the structure

$$\Lambda^{-6}\mathcal{F}_0^{(k)}(\mathcal{S}_i, \Lambda, \mu\Lambda) = \mathcal{H}^{(k)}\left(\frac{\mathcal{S}_1}{\Lambda^3}, \frac{\mathcal{S}_2}{\Lambda^3}\right) - \left(\frac{\mathcal{S}_1}{\Lambda^3} + \frac{\mathcal{S}_2}{\Lambda^3}\right)^2 \ln\mu\,. \tag{2.9}$$

Let us show that the dependence of S on Λ and Δ still follows after modifying the free energy as in (2.5). The extremum condition (2.3) holds unchanged also for the rescaled variables, in particular $\mathcal{S} \equiv \mathcal{S}_1 = e^{i\kappa}\mathcal{S}_2$. Set $\tau^{(k)} = \frac{\partial^2 \mathcal{F}_0^{(k)}}{\partial \mathcal{S}_i \partial \mathcal{S}_j}$. From the condition $\tau_{11}^{(k)} + \tau_{12}^{(k)} = 0$ we get the expansion of μ

$$\mu^4 = \frac{\mathcal{S}}{\Lambda^3} \exp\left[\sum_{n\geq 3} b_n \left(\frac{\mathcal{S}}{\Lambda^3}\right)^{n-2}\right], \tag{2.10}$$

where

$$b_n = \sum_{j=0}^{n} |c_{n,j}|(n-j)(n-2j-1).$$

We shall see that this expansion leads to a set of relations that constrain the coefficients of the free energy. Inverting (2.10) as a series for $\mathcal{S}/\Lambda^3$ in powers of μ^4, one obtains

$$\mathcal{S} = \Lambda^3(\mu^4 + 6\mu^8 + 140\mu^{12} + 4620\mu^{16} + \cdots), \tag{2.11}$$

that expressed in terms of S coincides with the series given in [17].

2.3 An unwanted term

The asymptotic expansion for $\tau^{(k)} \equiv \tau_{11}^{(k)}$ reads[1]

$$\tau^{(k)} = -\frac{\kappa}{2\pi} + \frac{1}{2\pi i}\ln\frac{\mathcal{S}}{\Lambda^3} + \frac{1}{4\pi i}\sum_{n\geq 3} n(n-1)a_n\left(\frac{\mathcal{S}}{\Lambda^3}\right)^{n-2}, \tag{2.12}$$

where $a_n = \sum_{j=0}^{n} |c_{n,j}|$. As a check of the formulation outlined so far, we would like to compare the result of the matrix model computation (2.12) to the known expression of the SW effective coupling constant τ.

In order to do this we have to plug the expansion (2.11) into (2.12). Since $\mu^4(u) = 2^{-6}(\Lambda_{SW}^2/u)^2$, where $\Lambda_{SW} = \sqrt{2}\Lambda$, by using the asymptotic expansion of $u(a)$ in [12] we find

$$\tau^{(k)} = -\frac{\kappa}{2\pi} + \tau_{SW}, \tag{2.13}$$

where the well known expression for the SW gauge coupling [12] reads, after setting $\hat{a} = a/\Lambda_{SW}$,

$$\tau_{SW} = \frac{2i}{\pi}\ln 2 + \frac{2i}{\pi}\ln\hat{a} + \frac{3}{4\pi i}\hat{a}^{-4} + \frac{105}{2^7\pi i}\hat{a}^{-8} + \frac{165}{2^7\pi i}\hat{a}^{-12} + \cdots. \tag{2.14}$$

Notice that the term $-\frac{\kappa}{4\pi}\mathcal{S}^2$ in the onshell rescaled free energy, which generates the discrepancy (2.13), cannot be reabsorbed by changing the phase of $\mathcal{S}$. Actually, the only phase that leaves the perturbative series of the onshell free energy $\sum_{n\geq 3} a_n(\mathcal{S}/\Lambda^3)^n$ invariant is $e^{2l\pi i}$, $l \in \mathbb{Z}$. On the other hand, we have

$$\mathcal{F}_0^{(k)}(e^{2l\pi i}\mathcal{S}) = \mathcal{F}_0^{(k-l)}(\mathcal{S}),$$

so that a term $\mathcal{S}^2$ multiplied by a half odd number would survive. The fact that $\tau^{(k)}$ does not exactly coincide with the SW effective coupling constant is a crucial question. Understanding and removing this discrepancy is a key step in our investigation.

[1] In the following expressions we have rescaled $\tau^{(k)}$ by $1/\pi i$.

2.4 Coincidences at the extremum

By evaluating the relevant quantities at the extremum, some interesting coincidences arise. The first step is a remark that, although obvious, needs to be stressed. This concerns how the prepotential is evaluated at the extremum. As we said, one first evaluates[2] τ at the extremum, then integrates it twice with respect to $\mathcal{S}$ (with care on the integration constants). Of course, this should be different from the function one obtains by directly evaluating it, that is $\mathcal{F}_0(\mathcal{S}_i, \Lambda, \mu\Lambda)$ with $\mathcal{S}_i$ and μ^4 replaced by their expressions at the extremum. We denote this function as

$$\hat{\mathcal{F}}_0(\mathcal{S}) \equiv \mathcal{F}_0|_{\mathcal{S},\mu}$$

where here and in the following we use the notation

$$f|_{\mathcal{S}} \equiv f|_{\mathcal{S}\equiv\mathcal{S}_1=e^{i\kappa}\mathcal{S}_2}\,, \qquad f|_{\mathcal{S},\mu} \equiv f|_{\mathcal{S}\equiv\mathcal{S}_1=e^{i\kappa}\mathcal{S}_2,\mu=\mu(\mathcal{S})}\,.$$

Remarkably, it turns out that directly evaluating $\mathcal{F}_0$ at the extremum one gets

$$\hat{\mathcal{F}}_0(\mathcal{S}) = 4\mathcal{F}_0(\mathcal{S})\,. \tag{2.15}$$

Let us set $\hat{\mathcal{S}}_D^i(\mathcal{S}) \equiv \frac{\partial\mathcal{F}_0}{\partial\mathcal{S}_i}|_{\mathcal{S},\mu}$. We have

$$\hat{\mathcal{S}}_D^2(\mathcal{S}) = -\hat{\mathcal{S}}_D^1(\mathcal{S}) = -2\mathcal{S}_D(\mathcal{S}) = -\frac{1}{2}\frac{\partial\hat{\mathcal{F}}_0}{\partial\mathcal{S}}\,, \tag{2.16}$$

where $\mathcal{S}_D(\mathcal{S}) = \frac{\partial\mathcal{F}_0(\mathcal{S})}{\partial\mathcal{S}}$. Since μ appears in $\mathcal{F}_0$ only through the term $-(\mathcal{S}_1+\mathcal{S}_2)^2\ln\mu$, it follows that in evaluating $\hat{\mathcal{F}}_0$ and $\hat{\mathcal{S}}_D^i$ we do not need the value of μ at the extremum (given in eq. (2.10)). In other words, just setting $\mathcal{S}\equiv\mathcal{S}_1 = e^{i\kappa}\mathcal{S}_2$, we obtain both $\hat{\mathcal{F}}_0$ and $\hat{\mathcal{S}}_D^i$. In order to evaluate $\mathcal{F}_0$ directly at the extremum we need only this "trivial part" of the condition coming from the extremum. In particular, we have

$$\tau = \frac{1}{2}\frac{\partial\hat{\mathcal{S}}_D}{\partial\mathcal{S}} = \frac{1}{4}\frac{\partial^2\hat{\mathcal{F}}_0}{\partial\mathcal{S}^2}\,.$$

Consider now the following nonperturbative relation [13]

$$\mu^4 = \frac{3\cdot 2^4\pi i}{\Lambda^6}\left(\mathcal{F}_0^{(k)} - \frac{\mathcal{S}}{2}\frac{\partial\mathcal{F}_0^{(k)}}{\partial\mathcal{S}}\right), \tag{2.17}$$

which is the analogue of the $\mathrm{U}(1)_R$ anomaly equation derived in SW theory [20]. A first interesting consequence of the above coincidences is that this relation between μ and the prepotential also holds, except for a factor 4, if one first computes the Legendre transform of $\mathcal{F}_0$ with respect to $\mathcal{S}_i^2$, and then evaluates it at the extremum. Since, as we said, the critical values are independent of the value of μ at the extremum, by (2.17) and (2.15) we obtain

$$\frac{\Lambda^6}{6\pi i}\mu^4 = (2\mathcal{F}_0 - \mathcal{S}_j\partial_{\mathcal{S}_j}\mathcal{F}_0)|_{\mathcal{S}} = 8\mathcal{F}_0 - 4\mathcal{S}\partial_{\mathcal{S}}\mathcal{F}_0 = 2\hat{\mathcal{F}}_0 - \hat{\mathcal{S}}_j\hat{\mathcal{S}}_D^j = 2\hat{\mathcal{F}}_0 - \hat{\mathcal{S}}\partial_{\mathcal{S}}\hat{\mathcal{F}}_0\,, \tag{2.18}$$

[2] In this subsection we omit the superscript k labelling the branches.

where $\hat{\mathcal{S}}_1 \equiv \mathcal{S}$, $\hat{\mathcal{S}}_2 \equiv -\mathcal{S}$. Among the various versions (2.18) of the relation (2.17), there is only one which can be satisfied by the unrescaled $\mathcal{F}_0(S_i)$, i.e.

$$u = \frac{3\pi i}{2\Lambda^4}\Big(2\mathcal{F}_0(S_i) - S_j\partial_{S_j}\mathcal{F}_0(S_i)\Big)|_{S,\mu}\,, \tag{2.19}$$

where $\mu = (\Lambda^2/4u)^{1/2}$. This is the version of the relation found in [13] in the form derived by Dymarsky and Pestun [16] (see also [21]). In this respect we note that while the relation between μ and $\mathcal{F}_0(\mathcal{S}_i, \Lambda, \mu\Lambda)$ holds in the versions given in (2.18), this is not the case for $\mathcal{F}_0(S_i, \Delta, \Lambda)$ that satisfies the relation only in the case in which the extremum is considered after the Legendre transform with respect to S_i^2 has been evaluated, that is eq. (2.19).

The detailed analysis of these coincidences will be presented elsewhere [19]. The origin of the observed coincidences relies on two crucial facts, namely the symmetry (2.6) of the free energy and the remarkable structure (2.9), that emerges after the rescaling. Moreover, due to the latter structure, one can easily prove that the gauge coupling is not affected by replacing the term $\ln\mu$ by a generic function $f(\mu)$, including $f \equiv 0$ (in this case $\tau_{ij} = \mathcal{H}_{ij}$).

2.5 Recursion relations

The relation between the $\mathcal{N} = 2$ u-modulus and the prepotential, in the context of the SW theory, leads to the proof of the SW conjecture [22]. The analogous relation in the matrix model is given by (2.17). Since this has a nonperturbative nature, it can then be argued that this relation puts strong constraints on the structure of the matrix model formulation itself. Remarkably, this is indeed the case as by (2.10) and (2.17) we get

$$\exp\left[\sum_{n\geq 3} b_n\left(\frac{\mathcal{S}}{\Lambda^3}\right)^{n-2}\right] = 1 - 6\frac{\mathcal{S}}{\Lambda^3} + 6\sum_{n\geq 3}(2-n)a_n\left(\frac{\mathcal{S}}{\Lambda^3}\right)^{n-1}\,, \tag{2.20}$$

which provides infinitely many conditions on the coefficients $c_{n,i}$ of the $\mathcal{N} = 1$ free energy. Even if apparently these conditions do not unequivocally fix the $c_{n,i}$ it is plausible that there exists a simple argument leading to fix them completely. In particular,

$$b_{n+3} - \frac{6n}{n+1}b_{n+2} + 6na_{n+2} - \frac{6}{n+1}\sum_{j=0}^{n-2}(j+1)(n-j-1)a_{n-j+1}b_{j+3} = 0\,.$$

which has been explicitly checked up to $n = 7$ [19].

3. Branching the matrix model

At this stage it is useful to summarize some questions one meets in the matrix model formulation of supersymmetric gauge theories. Even if we are considering the specific case of the CIV free energy [1], the issues we are dealing with extend to more general cases.

The first problem concerns the gauge coupling constant. A starting point of our investigation was (2.4) showing that $S_1 = -S_2$ is not a solution. On the other hand, this is related to the lack of symmetry of the original free energy (2.1) under $S_1 \to -S_2$, $S_2 \to -S_1$. We then saw that this symmetry, and therefore the solution $S_1 = e^{i\kappa}S_2$, can be restored by including additional terms to the free energy (2.1) depending on the odd number $\kappa/\pi = 2k+1$ which specifies the symmetry, namely

$$S_1 \longrightarrow e^{i\kappa} S_2 \,, \qquad S_2 \longrightarrow e^{-i\kappa} S_1 \,.$$

In this way one obtains the correct critical values for μ, $\mathcal{F}_0^{(k)}$ and therefore $\tau^{(k)}$. However, as we said, in comparing $\tau^{(k)}$ with the SW effective coupling constant, one sees that they coincide up to the term $-\frac{\kappa}{2\pi}$. This is not a minor question. First of all note that the exact expression of τ is necessary to get the correct monodromy. In general, rescaling or adding a constant to a function breaks its Möbius polymorphicity, in our case

$$\tau \longrightarrow \tilde{\tau} = \frac{A\tau + B}{C\tau + D} \,, \tag{3.1}$$

where the constants are the entries of the matrices in $\Gamma(2)$. The only possibility to add a constant by preserving the monodromy properties of τ is that such a constant corresponds to a translation in $\Gamma(2)$. On the other hand, $\tau^{(k)}$ in (2.12) differs from the SW effective coupling constant τ by the constant $-\frac{\kappa}{2\pi}$, that is

$$\tau^{(k)} = \tau - \frac{\kappa}{2\pi} \,, \tag{3.2}$$

and since the difference $\tau^{(k)} - \tau$ is a non integer number, (3.2) cannot correspond to a $\Gamma(2)$ monodromy of τ.

Another open question is related to the fact this formulation of the conjecture provides an expression for $\mathcal{F}_0^{(k)}(S_j, \Delta, \Lambda)$ while, in order to display the duality structure, we are forced to consider its rescaled version $\mathcal{F}_0^{(k)}(\mathcal{S}_j, \Lambda, \mu\Lambda) = \mu^6 \mathcal{F}_0^{(k)}(S_j, \Delta, \Lambda)$, as observed in [13]. The properties of $\mathcal{F}_0^{(k)}(\mathcal{S}_j, \Lambda, \mu\Lambda)$ indicate that this rescaling actually hides a property of the matrix model formulation which is still to be understood.

The last question concerns the nature of Δ, which is related to the scaling properties of the free energy. In the usual formulation, Δ is to be identified with $2\sqrt{u}$ by comparing the matrix model curve at the extremum with the SW curve. But one should be led to investigate the meaning of Δ even outside the critical point. On the other hand, in passing to the effective superpotential of the gauge theory we would like to consider the $\mathcal{N} = 1$ dinamically generated scale Λ_1 rather than Δ. Related to these questions is the unpleasant feature that, in order to derive the free energy, one is forced to identify dimensionless quantities with dimensionful ones.

The above list concerns the main, strictly connected, questions related to the matrix model formulation of supersymmetric gauge theories. The problem is to understand whether there exists an exact matrix model formulation free of the above problems. In the following we will see that in fact there exists such a formulation and that, while possessing some of the relevant features of the original formulation, it leads to a natural explanation based on the two different dynamically generated scales of the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ theories.

3.1 Branches in the matrix model

Let us consider as our starting point the matrix model with cubic potential [17, 23]. We now show that a suitable modification of that model actually leads to a formulation free of the problems outlined in the previous section. In order to find the matrix model potential we first note that we should introduce, in the usual cubic potential, the branches we labelled by the integer k

$$W^{(k)} = \mathrm{Tr}\,(m^{(k)}\Phi^2 + g^{(k)}\Phi^3), \qquad k \in \mathbb{Z}\,.$$

To specify the meaning of the index in the matrix potential we try to eliminate some of the dimensional problems one has from the very beginning in the formulation. First, since Φ are dimensionless quantities, in order to be consistent one should require that $m^{(k)}$ and $g^{(k)}$ be dimensionless quantities.

Eliminating dimensionful quantities from the potential leads us to consider the following dimensionless branched matrix model potential

$$W^{(k)}(\Phi) = \mathrm{Tr}\left(\frac{1}{2}e^{\frac{i}{2}\kappa}\Phi^2 + \frac{1}{3}e^{\frac{3i}{4}\kappa}\Phi^3 - \frac{\mathbf{1}}{12}\right), \tag{3.3}$$

$\kappa \equiv (2k+1)\pi, k \in \mathbb{Z}$, where we added an integration constant for future purpose. With respect to the previous formulation this potential does not contain any parameter. Even if surprising, we will show that this will reproduce the exact $\mathcal{N}=2$ results and will be free of the problems outlined so far. Note that the κ-dependence can be completely absorbed in the matrix redefinition $\Psi_k = e^{\frac{i}{4}\kappa}\Phi$ so that

$$W^{(k)}(\Phi) = \mathrm{Tr}\left(\frac{1}{2}\Psi_k^2 + \frac{1}{3}\Psi_k^3 - \frac{\mathbf{1}}{12}\right), \tag{3.4}$$

in other words

$$W^{(k)}(\Phi) = W^{(0)}(\Psi_k)\,.$$

We consider the two cut solution in which M_1 eigenvalues fluctuate around to the critical point $a_1 = 0$ and $M_2 = M - M_1$ eigenvalues fluctuate around the other critical point $a_2 = -e^{-\frac{i}{4}\kappa}$. As usual, one passes to the eigenvalue representation getting as jacobian the square of the Vandermonde determinant. In terms of the fluctuations around the two vacua

$$\lambda_i = a_1 + \nu_{1i}\,, \quad i = 1,\ldots,M_1\,, \qquad \lambda_i = a_2 + \nu_{2i}\,, \quad i = M_1+1,\ldots,M\,,$$

we can exponentiate the Vandermonde determinant and obtain the matrix model around this vacuum

$$Z^{(k)} = \frac{1}{\mathrm{Vol}(\mathrm{U}(M_1)) \times \mathrm{Vol}(\mathrm{U}(M_2))}\int D\Phi_1 D\Phi_2 \mathrm{e}^{-W_1^{(k)}(\Phi_1) - W_2^{(k)}(\Phi_2) - W_I^{(k)}(\Phi_1,\Phi_2)}\,, \tag{3.5}$$

where

$$\begin{aligned} W_1^{(k)} &= \mathrm{Tr}\left(\frac{1}{2}e^{\frac{i}{2}\kappa}\Phi_1^2 + \frac{1}{3}e^{\frac{3i}{4}\kappa}\Phi_1^3 - \frac{\mathbf{1}}{12}\right), \\ W_2^{(k)} &= -\,\mathrm{Tr}\left(\frac{1}{2}e^{\frac{i}{2}\kappa}\Phi_2^2 - \frac{1}{3}e^{\frac{3i}{4}\kappa}\Phi_2^3 - \frac{\mathbf{1}}{12}\right). \end{aligned}$$

Note that the constant term $-\mathbf{1}/12$ in (3.3) is the one which leads to a constant contribution to $W_1^{(k)} + W_2^{(k)}$ that vanishes when $M_1 = M_2$. The interaction term $W_I^{(k)}$ is obtained by expanding the log when exponentiating the Vandermonde determinant. While referring

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to [19] for the details about the evaluation of this matrix model, here we just comment on the quadratic contribution. This is important as it shows that whereas, as in the previous approaches, the propagator has the "wrong" sign, this is precisely what we need. While this is usually seen as a problem of the formulation and so its effect is essentially ignored, we see that the minus sign leads to the correct expression for the free energy. If m denotes the coefficient of the quadratic contribution, then it is usually assumed that this leads to $m^{-\frac{1}{2}(M_1^2+M_2^2)}$. However, in our case $m = e^{\frac{i}{2}\kappa}$ and the minus sign for the quadratic contribution to the second matrix potential corresponds to a minus sign of the exponent

$$-e^{\frac{i}{2}\kappa} = e^{-\frac{i}{2}\kappa}.$$

It follows that the quadratic terms give the following contribution to $Z^{(k)}$

$$e^{-\frac{i}{4}\kappa(M_1^2-M_2^2)},$$

which is exactly what we need for reproducing the second term in (2.5). So, we see that the minus sign turns out to be correct in the matrix model formulation. Finally, the planar contribution to the free energy reads

$$\begin{aligned} F_0^{(k)}(M_j) &= \frac{1}{12}(M_1 - M_2) + \sum_{i=1,2} \frac{M_i^2}{2} \ln M_i - \frac{i}{4}\kappa(M_1^2 - M_2^2) - \\ &\quad - \frac{i}{2}\kappa M_1 M_2 - \frac{3}{4}(M_1^2 + M_2^2) + \sum_{n\geq 3}\sum_{j=0}^{n} c_{n,j} M_1^{n-j} M_2^j . \end{aligned} \tag{3.6}$$

3.2 The gauge theory coupling

The above results would suggest identifying M_i with S_i/Δ^3. However, in this case the new expression (3.6), besides a global rescaling, displays two basic differences with respect to the old free energy (2.5).

First of all we note the absence of the term

$$(S_1 + S_2)^2 \ln\left(\frac{\Lambda}{\Delta}\right). \tag{3.7}$$

This term is problematic because, on the one hand, there is no reason for the appearence of Λ in the matrix model free energy; as we already pointed out, the Λ dependence is usually added by hand from a gauge theory guess. On the other hand, we have seen that this term plays a basic role in evaluating the extremum of the effective superpotential.

The second difference is that whereas in $\mathcal{F}_0^{(k)} = \mathcal{F}_0 + \delta\mathcal{F}_0^{(k)}$, as in (2.5), we have the term

$$\frac{i}{2}\kappa S_1 S_2, \tag{3.8}$$

the analogous contribution in (3.6), that is $-\frac{i}{2}\kappa M_1 M_2$, has opposite sign. Fortunately, we saw that directly evaluating the rescaled free energy $\mathcal{F}_0^{(k)}(\mathcal{S}_1, \mathcal{S}_2)$ at $\mathcal{S}_1 = e^{i\kappa}\mathcal{S}_2$, reproduces the SW prepotential except for the term $-(2k+1)\mathcal{S}^2$. It is precisely because of this change of sign of the term (3.8) in (3.6), that this unwanted additional term is now missing, so

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that the first condition coming from the extremum, i.e. $\tau_{11}^{(k)} = \tau_{22}^{(k)}$, by itself reproduces the correct SW coupling as in (2.14). But we should now take into account both the change of sign of (3.8) and simultaneously get the exact expression for μ. Nevertheless, requiring that the second condition be consistent with the first one, leads to a new view on the structure of the bare coupling constant τ_0! The idea is quite natural. Namely, note that in evaluating the extremum (2.3) we have forgotten the bare coupling constant by simply putting it to zero. But we can just use a nonzero value of τ_0 to get the correct critical values. More precisely, we set

$$\tau_0 \longrightarrow \tau_0^{(k)} = \frac{2}{\pi i} \ln \frac{\Lambda_2}{\Lambda_1} - \frac{\kappa}{2\pi} \,. \tag{3.9}$$

Before identifying the two scales Λ_1 and Λ_2, we make a couple of comments on this proposal. A first interesting consequence of (3.9) is that

$$\tau_0^{(j)} - \tau_0^{(k)} = k - j \,, \tag{3.10}$$

that will be discussed in [19]. The second observation is that, as we will see, the term $\frac{2}{\pi i} \ln \frac{\Lambda_2}{\Lambda_1}$ inside $\tau_0^{(k)}$ compensates the fact that (3.7) is now missing, as it should, since it cannot derive from the matrix model, from the new free energy (3.6). Furthermore, the role of the term $\frac{\kappa}{2\pi}$ in $\tau_0^{(k)}$ is that of compensating the change of sign of the term (3.8) in (3.6), a request coming from the need of obtaining the correct expression for μ as given in (2.10).

Let us now identify the scales Λ_1 and Λ_2. The meaning of Λ_2 is obvious, as it plays the role of the scale Λ appearing in the expression of the $\mathcal{N} = 2$ effective coupling, so that $\Lambda_2 \equiv 2^{-1/2}\Lambda_{SW}$. Since in this new approach the Δ parameter simply disappeared, the natural choice for Λ_1 is just the $\mathcal{N} = 1$ dynamically generated scale, as it should appear in the expression of the effective potential. Therefore, we have

$$\Lambda_1 \equiv \Lambda_{\mathcal{N}=1} \,, \qquad \Lambda_2 \equiv \Lambda_{\mathcal{N}=2} \,.$$

3.3 The prescription

We now consider the link between the matrix model and the gauge theory. The prescription is to make the following dimensionless identification in (3.6)

$$M_1 = \frac{S_1}{\Lambda_1^3} \,, \qquad M_2 = \frac{S_2}{\Lambda_1^3} \,, \tag{3.11}$$

with the free energy given by

$$\mathcal{F}_0^{(k)}(S_i, \Lambda_1) = \Lambda_1^6 F_0^{(k)} \left(\frac{S_i}{\Lambda_1^3} \right) , \tag{3.12}$$

that is

$$\begin{aligned} \mathcal{F}_0^{(k)}(S_i, \Lambda_1) = {} & \frac{\Lambda_1^3}{12}(S_1 - S_2) + \sum_{j=1,2} \frac{S_j^2}{2} \ln \left(\frac{S_j}{\Lambda_1^3} \right) - \frac{i}{4}\kappa(S_1^2 - S_2^2) - \\ & - \frac{i}{2}\kappa S_1 S_2 - \frac{3}{4}(S_1^2 + S_2^2) + \Lambda_1^6 \sum_{n\geq 3} \sum_{j=0}^{n} c_{n,j} \left(\frac{S_1}{\Lambda_1^3} \right)^{n-j} \left(\frac{S_2}{\Lambda_1^3} \right)^{j} . \end{aligned} \tag{3.13}$$

Note that in the present derivation the M_i are identified with the dimensionless quantities S_i/Λ_1^3 and there is no need to add by hand any additional scale. Furthermore, we note that one can consistently define the matrix model with M_i replaced by S_i/Λ_1^3. After this identification is made, one analytically continues S_i/Λ_1^3 so that the critical case can be consistently considered.

By (3.13) and (3.9) the gauge theory effective superpotential is

$$\begin{aligned} W_{\rm eff}^{(k)}(S_i) &= \sum_{j=1,2}\left(\frac{\partial\mathcal{F}_0^{(k)}}{\partial S_j} - 2\pi i\tau_0^{(k)}S_j\right) \\ &= \sum_{j=1,2} S_j\left[\ln\left(\frac{S_j}{\Lambda_1^3}\right) - 1 - 4\ln\left(\frac{\Lambda_2}{\Lambda_1}\right)\right] + i\kappa S_2 + \Lambda_1^3\sum_{n\geq 2}\sum_{j=0}^{n} d_{n,j}\left(\frac{S_1}{\Lambda_1^3}\right)^{n-j}\left(\frac{S_2}{\Lambda_1^3}\right)^{j}, \end{aligned} \tag{3.14}$$

where $d_{n,j} = c_{n+1,j}(n+1-j) + c_{n+1,j+1}(j+1)$. Observe that

$$W_{\rm eff}^{(k)}(e^{2k_i\pi i}S_i) = W_{\rm eff}^{(k)}(S_i) + \sum_{j=1,2} 2\pi i k_i S_i\,, \tag{3.15}$$

where $k_i \in \mathbb{Z}$, $i = 1,2$. It follows that the linear contribution to the effective superpotential has the structure $\sum_{j=1,2}(2\pi i k_j - 1)S_j + \pi i S_2$, so that the term $\pi i S_2$ plays a special role as it cannot be completely reabsorbed by a phase shift of the S_i.

Minimizing $W_{\rm eff}^{(k)}$ in (3.14) gives the two F-term conditions

$$S \equiv S_1 = e^{i\kappa}S_2\,, \tag{3.16}$$

and

$$\left(\frac{\Lambda_2}{\Lambda_1}\right)^4 = \frac{S}{\Lambda_1^3}\exp\Big[\sum_{n\geq 3} b_n\left(\frac{S}{\Lambda_1^3}\right)^{n-2}\Big]\,. \tag{3.17}$$

In order to identify the $\mathcal{N}=2$ effective coupling constant, we should first recognize the relationship between the two scales Λ_1 and Λ_2 at the extremum.

Additional contributions to the Veneziano-Yankielowicz superpotential [18] have already been considered in literature, for example by Kovner and Shifman [24]. More recently, Cachazo, Seiberg and Witten [7] first observed that in considering the critical points, which follow from the Veneziano-Yankielowicz superpotential W^{VY}, one may consider, in the case of SU(N), either $\ln(\Lambda^{3N}/S^N) = 0$ or $N\ln(\Lambda^3/S) = 0$, leading to an apparent ambiguity. Then they observed that W^{VY} can be defined on each of the N possible infinite cover of the S-plane. In particular, according to their analysis, one should explicitly include additional branches to W^{VY}. In the case of symmetry breaking pattern $\mathrm{U}(N) \to \prod_j \mathrm{U}(N_j)$ they obtained

$$W_{\rm eff}(S_i) = \sum_{j=1}^{n}\left(2\pi i\tau_0 S_j + N_j S_j\left[\ln\left(\frac{\Lambda_j^3}{S_j}\right) + 1\right] + 2\pi i b_j S_j\right) + \mathcal{O}(S_iS_j)\,, \tag{3.18}$$

where the b_j are integers with $b_1 = 0$. It is interesting to observe that the additional contribution $2\pi i\sum_j b_j S_j$, representing the relative shifts of theta vacua, is reminiscent of the $i\kappa S_2$ term in (3.14). However, note that whereas κ/π is odd, the additional term in (3.18) always depends on the even numbers $2b_j$. Furthermore, unlike (3.14), for each $N_j = 1$, the corresponding term $2\pi i b_j S_j$ in (3.18) can be exactly obtained, as it should, by the phase shift $S_j \to e^{2\pi i b_j} S_j$ in the argument of W_{eff}. The reason is that the b_j's label the theta vacua of each factor in the broken gauge group, and so they play no role in the abelian case. Therefore, even if these contributions have a similar structure, they appear of different nature. In particular, whereas the Cachazo, Seiberg and Witten term is based on the general properties of the logarithm, our additional term is a consequence of the request that at the critical point $S_1 + S_2 = 0$, as it should in the model we are considering. Nevertheless, in spite of the differences, it is likely that further investigation in this direction may lead to a better understanding of the Veneziano-Yankielowicz superpotential and related issues.

3.4 SW modulus and the scales

Apparently, we do not have any information concerning the identification of the u modulus. In previous formulations this was argued by identifying the parameters of the matrix model potential and the SW curve. Here we have a different view which is strictly related, as the structure of the bare coupling constant $\tau_0^{(k)}$ indicates, to the RGE. In the previous approaches one identified u in terms of Δ using the derivation of the matrix potential from the SW curve, leading to some questions outlined in previous sections. In our case, by (3.17) and the nonperturbative relation (2.17), which still holds for the free energy (3.13), we recover again the identity (2.20), as explained in detail in [19]. Therefore, by means of monodromy arguments we can make the identification at the critical point

$$\frac{\mathcal{S}}{\Lambda_2^3} = \frac{S}{\Lambda_1^3} \,. \tag{3.19}$$

Furthermore, this implies that the left hand side of (2.10) and that of (3.17) coincide, i.e.

$$\left(\frac{\Lambda_2}{\Lambda_1}\right)^4 = \left(\frac{\Lambda_2^2}{4u}\right)^2 ,$$

by means of which we recover the relation between the u-modulus and the $\mathcal{N} = 1$ scale

$$\Lambda_1^2 = 4u \,, \tag{3.20}$$

and (3.19) becomes $\mathcal{S}/\Lambda_2^3 = S/8u^{3/2}$. Thus we have found that in the present formulation at the critical point the $\mathcal{N} = 1$ scale coincides with 2 times the square root of the u-modulus of the $\mathcal{N} = 2$ theory.

We can as well modify the prescription (3.11) by means of (3.19) and make the following identification in (3.6)

$$M_1 = \frac{\mathcal{S}_1}{\Lambda_2^3} \,, \qquad M_2 = \frac{\mathcal{S}_2}{\Lambda_2^3} \,, \tag{3.21}$$

with the free energy given by

$$\mathcal{F}_0^{(k)}(\mathcal{S}_i, \Lambda_2) = \Lambda_2^6 F_0^{(k)}\left(\frac{\mathcal{S}_i}{\Lambda_2^3}\right) . \tag{3.22}$$

By evaluating $\mathcal{F}_0^{(k)}(\mathcal{S}_i, \Lambda_2)$ directly at $\mathcal{S} \equiv \mathcal{S}_1 = e^{i\kappa}\mathcal{S}_2$, we see that the k-dependence completely disappears. In particular, we obtain

$$\frac{1}{4\pi i}\mathcal{F}_0^{(k)}(\mathcal{S}, e^{-i\kappa}\mathcal{S}, \Lambda_2) = \frac{\Lambda_2^3}{24\pi i}\mathcal{S} - \frac{3}{8\pi i}\mathcal{S}^2 + \frac{1}{4\pi i}\mathcal{S}^2 \ln\frac{\mathcal{S}}{\Lambda_2^3} + \frac{\Lambda_2^6}{4\pi i}\sum_{n\geq 3} a_n \left(\frac{\mathcal{S}}{\Lambda_2^3}\right)^n , \tag{3.23}$$

that now precisely corresponds to the SW prepotential as obtained by integrating twice τ with respect to the glueball superfield.

Note that the absence of any parameter in the expression of the free energy allows us to look for its monodromy properties. To understand this aspect, observe that if the term $(S_1^2 + S_2^2)\ln\mu$ is present in the expression of the free energy, the monodromy would involve μ dependent terms leading to a rather involved analysis.

In the usual formulation the potential depends on some parameters, namely the couplings, whereas in (3.3) they are missing. This is due to the fact that simply we need not double the number of parameters. Actually, once S_1, S_2 and $\mathcal{F}_0$ are given, we have enough information to get the full SW theory. In particular, the above discussion shows that the u-modulus arises in terms of a through the relation (2.17).

4. Triangulating the instanton moduli space

Results in noncritical strings uncovered a deep connection between algebraic-geometrical structure and Liouville theory. It should be stressed that, on one hand, Liouville theory arises in the description of the moduli space of Riemann surfaces, in particular the Liouville action is the Kähler potential for the Weil-Petterson metrics. On the other hand, Liouville theory is the crucial quantum field theory for noncritical strings. In particular, in [25] it was shown that there is an analytic formulation for 2D pure quantum gravity which is directly expressed in terms of the Liouville geometry of moduli space of punctured spheres, reproducing the Painlevé I (Liouville F-models). In that paper it was also argued that the eigenvalues of the matrix model should be seen as punctures on a Riemann sphere, which can be identified with the branch points on the Riemann sphere itself. We note, in passing, that the relation between punctured spheres and hyperellittic Riemann surfaces leads to relations between Weil-Petterson volumes for such surfaces, e.g. $Vol_{WP}(\overline{\mathcal{M}}_{1,1}) = 2Vol_{WP}(\overline{\mathcal{M}}_{0,4})$. Furthermore, there is the isomorphism $\overline{\mathcal{M}}_{2,0} \cong \overline{\mathcal{M}}_{0,6}$. One may expect that these relationships hide more general properties of moduli spaces, which should be strictly related to the Deligne-Knudsen-Mumford compactification. The latter, together with the Wolpert restriction phenomenon, is at the heart of the recursion relations associated to the Painlevé I as derived in [25]. The analogy with the recursion relation for the $\mathcal{N} = 2$ instantons suggested the formulation of instanton numbers in terms of intersection theory [26]. Recalling that 2D quantum gravity also leads to a natural triangulation of moduli space of Riemann surfaces, one might expect that a similar structure arises in instanton theory. Remarkably, we have that the coincidences discussed in section 2 provide the following direct identification of the $\mathcal{N} = 2$ prepotential

$$e^{\hat{\mathcal{F}}/\Lambda_2^6} = \frac{1}{\mathrm{Vol}(\mathrm{U}(M_1)) \times \mathrm{Vol}(\mathrm{U}(M_2))}\int \mathcal{D}\Phi_1\mathcal{D}\Phi_2 e^{-W^{(k)}(\Phi_1,\Phi_2)}|_{M_1=e^{i\kappa}M_2\equiv\mathcal{S}/\Lambda_2^3} , \tag{4.1}$$

where

$$W^{(k)}(\Phi_1, \Phi_2) = W_1^{(k)}(\Phi_1) + W_2^{(k)}(\Phi_2) + W_I^{(k)}(\Phi_1, \Phi_2)\,.$$

This gives a direct way of expressing $\mathcal{N} = 2$ instanton contributions, including the gravitational corrections considered in [27, 28, 29, 23], in terms of a matrix model. The above remarks then suggest that this formulation should be related to a kind of triangulation of instanton moduli space.

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